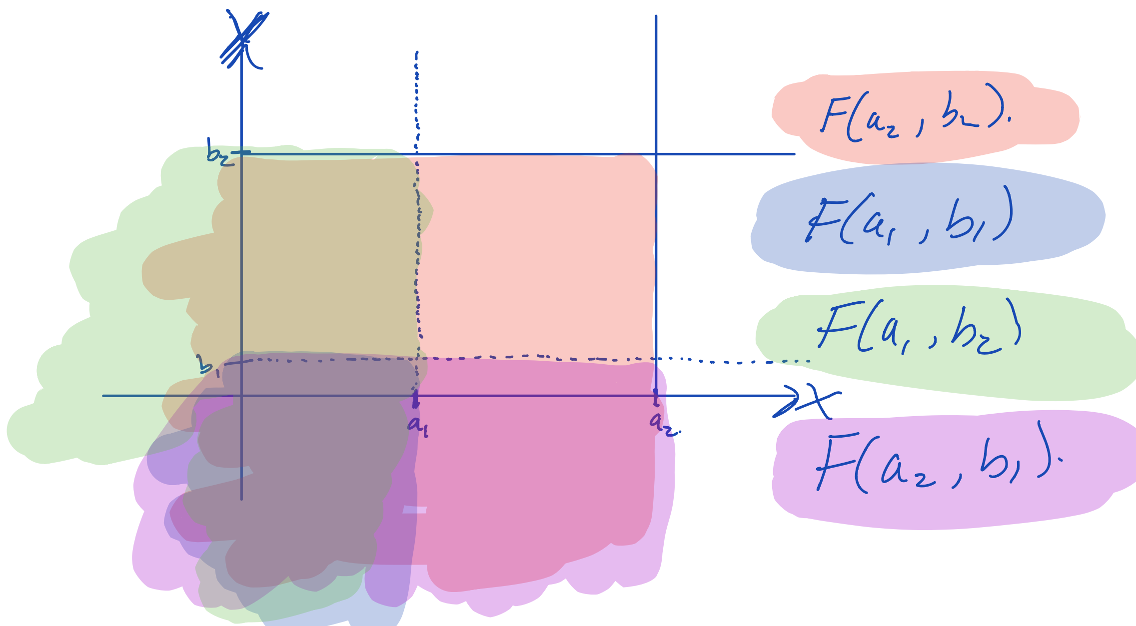
Lecture 24

- So far, we have been concerned with statements about single random variables.
- In this chapter we want to address questions about multiple random variables.
- For any two random variables X and Y, we define the Joint Cumulative Probability Distribution Function as

$$F(a,b) = P(X \le a, Y \le b), \quad -\infty < a, b < \infty.$$

- many questions about X and Y can be answered by $F(x,y)$. For example: suppose we want

$$P(a_1 < X \le a_2,\ b_1 < X \le b_2) = \cdots$$



$$= F(a_2,b_2) \underbrace{- F(a_1,b_2) - F(a_2,b_1)}_{\text{subtracted } F(a_1,b_1) \text{ twice (double counting)}} + \underbrace{F(a_1,b_1)}_{\text{adding back.}}$$

When X and Y are discrete, we define the <u>joint</u> <u>probability</u> mass function as

$$p(x,y) = P(X=x, Y=y).$$

Now, the events $\{\{X=x, Y=y_j\}, j\}$ are mutually disjoint, and so

$$\begin{aligned} p_X(x) &= P(X=x) \\ &= P\left(\bigcup_j \{X=x, Y=y_j\}\right) \\ &= \sum_j P\{X=x, Y=y_j\} \\ &= \sum_j p(x, y_j). \end{aligned}$$

← probability mass function of X as a random var on its own.

similarly $\quad p_Y(y) = \sum_i p(x_i, y)$

<u>Ex</u>: Suppose an urn consists of 3 red, 4 white, and 5 blue balls. Suppose we draw three balls. Let X be # of red balls, Y = # of white balls.

Let $p(i,j) = P(X=i, Y=j)$. Note that if of the three balls chosen i are red and j are white, then the remaining $3-i-j$ balls are blue. Since each draw is equally likely,

$$p(i,j) = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-i-j}}{\binom{12}{3}}.$$

<u>Rmk</u>: $P_Y(y)$ and $P_X(x)$ are also called <u>marginal probabilities</u>...

We say that X and Y are jointly continuous if there exist a function $f(x,y)$ such that for every (measurable) set $A \subseteq \mathbb{R}^2$, we have that

$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy.$$

We call $f(x,y)$ the "joint probability density function" of X and Y. If $A = C\times D$ is a rectangular region, then

$$P((X,Y)\in C\times D) = \int_D\int_C f(x,y)\,dx\,dy.$$

In particular, if we let $C=(-\infty,a]$, $D=(-\infty,b]$, then

$$F(a,b) = P((X,Y)\in(-\infty,a]\times(-\infty,b]) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

Differentiating we have

$$f(a,b) = \frac{\partial^2}{\partial a\,\partial b}F(a,b).$$

If X and Y are jointly continuous, then they are also continuous random variables on their own. We can obtain their (marginal) probability density functions as:

$$\begin{aligned}P(X\in A) &= P(X\in A,\ Y\in(-\infty,\infty))\\ &= \int_A\int_{-\infty}^{\infty} f(x,y)\,dy\,dx\\ &= \int_A f_X(x)\,dx.\end{aligned}$$

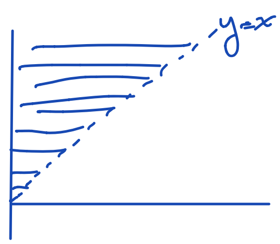
where $$f_X(x) = \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}_{\text{probability density function of } X.}$$

Similarly $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Ex: Suppose that
$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

a) Compute $P(X>1, Y<1)$.
$$P(X>1, Y<1) = \int_0^1\int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy = 2\left(\int_0^1 e^{-2y}\,dy\right)\left(\int_1^\infty e^{-x}\,dx\right)$$
$$= e^{-1}(1-e^{-2}).$$

b) $$P(X<Y) = \iint_{\{(x,y):\, x<y\}} 2e^{-x}e^{-2y}\,dx\,dy.$$



$$= \int_0^\infty\int_0^y 2e^{-x}e^{-2y}\,dx\,dy. = 1/3.$$

c) $$P(X<a) = \int_0^a\int_0^\infty 2e^{-2y}e^{-x}\,dy\,dx = 1-e^{-a}$$

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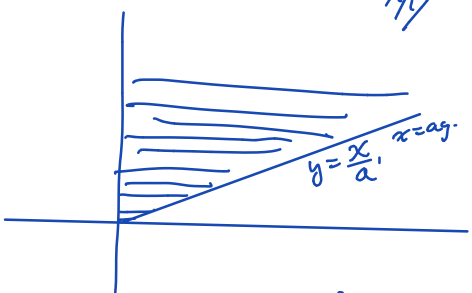
ex: Suppose that the joint density of X and Y is given by
$$f(x,y) = \begin{cases} e^{-(x+y)} & 0<x<\infty,\ 0<y<\infty. \\ 0 & \text{Otherwise.} \end{cases}$$

Find the density function of X/Y.

Soln: First, we find the cumulative distribution function of X/Y: For $a>0$, we have

$$F_{X/Y}(a) = P\left\{\frac{X}{Y} \le a\right\}.$$



$\{(x,y): \frac{x}{y} \le a\}$
$\parallel$
$\{(x,y): y \geq \frac{x}{a}\}$

$$= \iint_{x/y \le a} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy.$$

$$= \int_0^\infty e^{-y} \int_0^{ay} e^{-x}\,dx\,dy$$

$$= \int_0^\infty e^{-y}\left(1 - e^{-ay}\right) dy.$$

$$= -e^{-y} + \frac{e^{-y(a+1)}}{a+1}\Big|_0^\infty$$

$$= 1 - \frac{1}{a+1}$$

Differentiating, we get $f_{X/Y}(a) = \frac{1}{(a+1)^2}$

We can also define the joint probability of n random variables:

$$F(a_1, \dots, a_n) := P(X_1 \le a_1, \dots, X_n \le a_n).$$

$X_1, \dots, X_n$ are said to be jointly continuous if there is a function $f(x_1, \dots, x_n)$ such that for some set $C \subseteq \mathbb{R}^n$,

$$P((X_1, \dots, X_n) \in C) = \iint_C \cdots \int f(x_1, \dots, x_n)\, dx_1, \dots, dx_n.$$

Ex: Suppose we have an experiment with r possible outcomes, with the ith outcome occuring with probability p_i, each mutually exclusive, $\sum_{i=1}^{r} p_i = 1$. Suppose that we perform the experiment n times. Let X_i be the number of experiments with outcome i. Then the joint distribution

$$P(X_1 = n_1, \dots, X_r = n_r) = \frac{n!}{n_1! \cdots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

when $\sum n_i = n$, is called the multinomial distribution.

When $r=2$, then we have the binomial distribution.